A Discrete Approach to Continuous Logistic Growth

Dan Kalman
American University
Washington, D.C. 20016
kalman@american.edu

Abstract: The development and application of mathematical models is a common component in the prior-to-calculus curriculum, and logistic growth is often considered in that context. A discrete version of logistic growth, based on a difference equation, provides a nice case study of model development and refinement. It allows students to understand how such models arise, and using numerical methods, how they can be applied. The continuous version of logistic growth is mathematically better behaved, and amenable to analytic methods accessible to precalculus students, but more difficult to motivate or derive. In this paper we discuss an approach to discrete logistic growth that leads in a natural and accessible way to continuous logistic growth functions.

The development and application of mathematical models is a common component in the prior-to-calculus curriculum (hereafter simply *precalculus*)[1], and logistic growth is often considered in that context. Conceptually, logistic growth can be understood as a refinement of exponential growth. Whereas exponential growth is unlimited and accelerating, we expect population growth to be limited. This difference is depicted graphically in Figure 1.

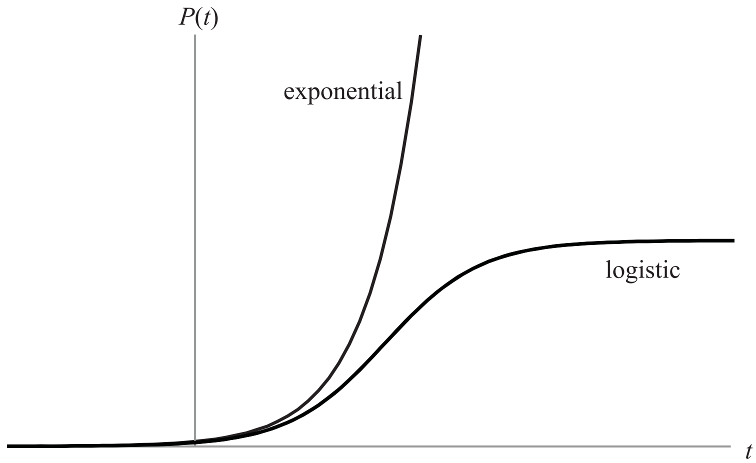

Figure 1: Exponential and logistic growth curves.

Both discrete and continuous models for logistic growth appear in textbooks and on-line references. The discrete approach is discussed in [9, 12, 14, 18]. It is typically introduced via a difference equation of the form (or equivalent to)

$$p_{n+1} = k(L - p_n)p_n, \tag{1}$$

where k and L represent numerical parameters, and p_n is the size of the population after n discrete time periods, say years, months, or reproductive cycles. Starting from an initial population of size p_0, we may use (1) recursively to compute successive terms of the sequence $\{p_n\}$.

The discrete approach has the advantage that it is easily motivated using ideas accessible to a precalculus audience. These models are conveniently explored numerically

[1] We distinguish between the course that immediately precedes calculus, often titled *Precalculus*, and a broader category of classes that precede calculus, such as *College Algebra* and foundational college classes at a similar level. In this paper, the term *precalculus* will have the broader meaning.

with a hand calculator or computer spreadsheet software. And they can produce very attractive and appealing graphs demonstrating the ideas of limited and sustainable growth, quite like the logistic curve in Figure 1.

On the other hand, (1) has the disadvantage that no general formula expresses p_n as a function of n. This restricts the analytic methods students can apply. For example, there is no convenient analytic method to predict values of p_n for given values of n; nor to discover when p_n achieves or surpasses a specified value. An equation for p_n as a function of n is a powerful and familiar tool for analyzing *end behavior*, and thus showing that p_n approaches an equilibrium. Lacking such an equation makes this task more difficult.

Discrete logistic growth models can also exhibit quite pathological behavior, at least from the perspective of a population model. In fact, it is famous as an example of a simple discrete model with extremely complicated (and sometimes chaotic) dynamics. In such cases, even numerical exploration becomes impractical. For all of these reasons, discrete logistic growth models are an excellent topic for emphasizing how models are formulated, but are not as useful in illustrating the analysis of such models.

In contrast to the discrete case, a continuous logistic growth model predicts the population at time t by an explicit function of the form

$$P(t) = \frac{A}{1 + Bb^t}. \tag{2}$$

This topic is considered in many precalculus texts, including [5, 7, 9, 13, 19, 23]. When the constants A, B, and b are positive, and $b < 1$, the graph of such function is a beautiful smooth curve very similar to what appears in Figure 1. There is a unique inflection point about which the graph is symmetric. Using precalculus techniques, students can evaluate $P(t)$, solve for t, locate asymptotes, and find the unique logistic curve for a specified inflection point and initial value $P(0)$. By estimating the inflection point and $P(0)$ from a graph of data points, it is therefore possible to define an approximating logistic curve.

On the other hand, although these curves arise very naturally in connection with elementary differential equations, they are difficult to motivate for students who have not studied calculus. Typically, (2) is simply proclaimed to be a logistic function, and its properties are presented, with little or no explanation of how this particular family of functions was first formulated. For example, Crauder, Evans, and Noell give a conceptual description of limited population growth, and then state *Deriving the formula for logistic growth requires techniques beyond the scope of this text, and we simply state the result.* [5, p. 319]. As these observations show, continuous logistic curves offer a rich example of how mathematical models can be analyzed, but are not well adapted to illustrating how such models are formulated.

The goal of this paper is to present a connection between discrete and continuous logistic growth models that combines the best pedagogical features of each. This approach uses ideas and methods that are as accessible as those of the discrete version of logistic growth, but lead in a natural way to a refined discrete logistic growth model, and to continuous logistic growth curves. In particular, I will show that the derivation of (2) need not be considered beyond the scope of a text at the level of College Algebra or Precalculus. The development also illustrates an important aspect of modeling: reconsidering simplifying assumptions in light of the results they produce.

Much of what is presented here can be directly adapted for classroom use. But I will also mention some material that would not be suitable for students. This includes a few mathematical morsels that are very close to the main narative thread – too close to pass by without comment – as well as some historical background for the refined discrete logistic model we will consider.

The paper is organized as follows. We proceed next to consider the discrete version of logistic growth in greater detail. We will trace the rationale behind the model, and numerically explore several examples. That will lead to the reconsideration of the model, and adoption of a revised difference equation. The behavior of the revised model will be explored numerically, and an equation for p_n as a function of n will be derived. Indeed, we will see that this equation has the exact form of a continuous logistic curve. Finally, we will touch briefly on several interesting connections to other areas of mathematics, and on the history of the revised difference equation.

Discrete Logistic Growth

In developing a population model, a constant relative growth rate is a natural first assumption. In a discrete model, this rate would represent growth over time intervals of some fixed length. Consider as an example the growth of a population of bacteria. Assuming that a fixed percentage of the population reproduces each hour leads to an equation of the form

$$p_{n+1} = rp_n \tag{3}$$

where p_n is the population at the start of hour n, and r is a fixed growth factor. Here, each term p_n is multiplied by r to produce the next term, and we immediately see that the sequence will be

$$p_0, p_0r, p_0r^2, \cdots.$$

Thus $p_n = p_0r^n$ holds for any natural number n. This equation expresses p_n explicitly as a function of n, and is called a *solution* of the difference equation (3). The factor r exceeds unity by the relative growth rate. For example, if the population increases by 6% each hour then $r = 1.06$. In general, with a positive relative growth rate, the solution to (3) is an exponential function with base $r > 1$.

Although this population model might be fairly accurate in the short term, we recognize that exponential growth cannot be an accurate long term model. A real population will be limited by environmental factors, and presumably subject to an absolute upper bound. Seeking to refine our model, it is reasonable to make r vary with the population size. The difference equation would then be

$$p_{n+1} = r(p_n)p_n \tag{4}$$

where the function $r(p)$ decreases as the population p increases. From a mathematical standpoint, it is natural to begin an investigation of such models by considering the case of a linear function $r(p)$.

Example 1. Let us look at a specific example. We can formulate an equation for $r(p)$ by making some simple assumptions about the characteristics of the population. We envision a population that initially grows with very little environmental constraint, a situation that might arise when a few members of a new species are introduced into an environment rich in nutrients and habitable area. Over time the population will increase until it approaches some maximum sustainable size, eventually reaching an equilibrium. At that stable population size, the growth factor should equal one.

Introducing specific numbers, suppose that a population of $p = 1000$ experiences a growth factor of $r(1000) = 1.4$. Assume also that the equilibrium population size is 5000, so that the corresponding growth factor is $r(5000) = 1$. The linear function r is thus determined to be

$$r(p) = 1.5 - .0001p,$$

and (4) becomes

$$p_{n+1} = (1.5 - .0001p_n)p_n.$$

This is an instance of discrete logistic growth. Rewriting the equation as

$$p_{n+1} = .0001(15000 - p_n)p_n,$$

we see that it has the same form as (1).

Unlike the case of geometric growth, in general there is no explicit solution to a discrete logistic growth difference equation. That is, we can not express p_n as a function of n using elementary functions. But the behavior of the sequence is readily studied numerically and graphically. Starting with an initial population $p_0 = 200$, we apply the difference equation repeatedly to determine several terms of the sequence p_n. This produces the table and graph shown in Figure 2.

n	p_n
0	200.0
1	296.0
2	435.2
3	633.9
4	910.7
5	1283.1
6	1760.0
7	2330.3
8	2952.4
9	3556.9
10	4070.2
11	4448.6
12	4693.9
13	4837.6
14	4916.2
15	4957.4

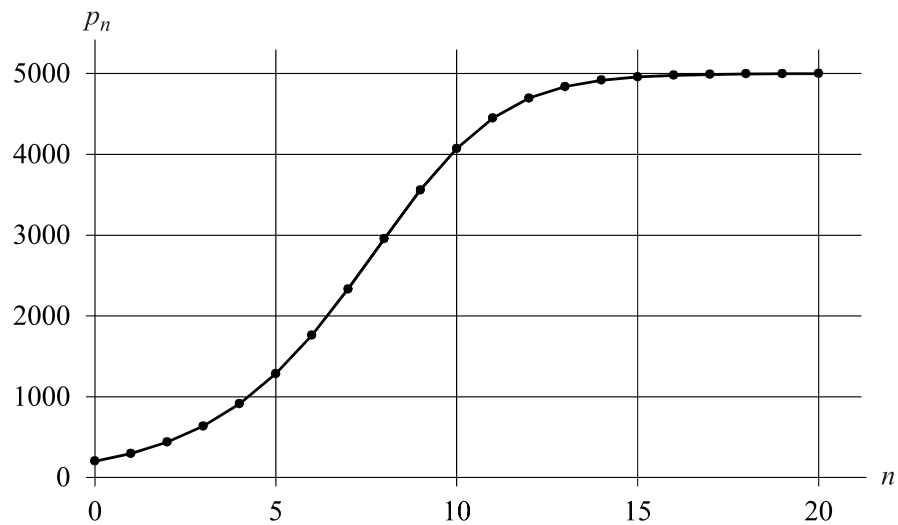

Figure 2: Coordinates and graph for Example 1. The table shows n and p_n values produced by the difference equation. The graph shows the points (n, p_n), connected by straight lines for visual emphasis.

This seems to be a very plausible model for limited growth. Initially the population increases rapidly, but then the growth rate declines and the population approaches an equilibrium. Visually, the graph closely resembles a continuous logistic growth curve. Although the development so far has not depended on any actual data, the results are encouraging.

But discrete logistic growth is well known as an example of an apparently simple model with potentially complicated dynamics. Some additional examples will illustrate this point.

Example 2. We repeat essentially the same analysis as in Example 1, but with a minor change in the parameters. This time, we suppose that a population of $p = 1000$ experiences a growth factor of $r(1000) = 2.7$, and that the equilibrium population size is 6000. Thus we have two points, $(p, r) = (1000, 2.7)$ and $(6000, 1)$, and find the linear function

$$r(p) = 3.04 - .00034p.$$

This leads to the difference equation

$$p_{n+1} = (3.04 - .00034p_n)p_n,$$

and starting again with an initial population size of 200, we can compute and graph several terms of the sequence. The resulting graph is shown in Figure 3.

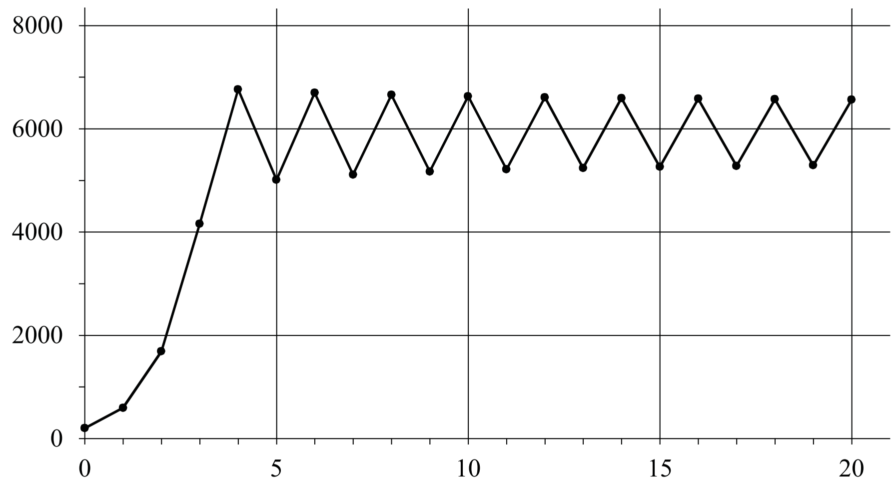

Figure 3: Graph for Example 2.

This is not nearly as attractive as the first example. According to this model, the population does not sedately approach an equilibrium, but rather settles into a persistent oscillation between two fixed population sizes. Although the basic form of the difference equation is the same as for Example 1, and the numerical values of the parameters are not so very different, the dynamics of the two examples are completely different.

Example 3. Modifying the preceding example again, we define $r(1000) = 3.1$, while leaving the equilibrium population size as 6000. Proceeding as before we derive the difference equation

$$p_{n+1} = (3.52 - .00042p_n)p_n,$$

and generate the graph shown in Figure 4.

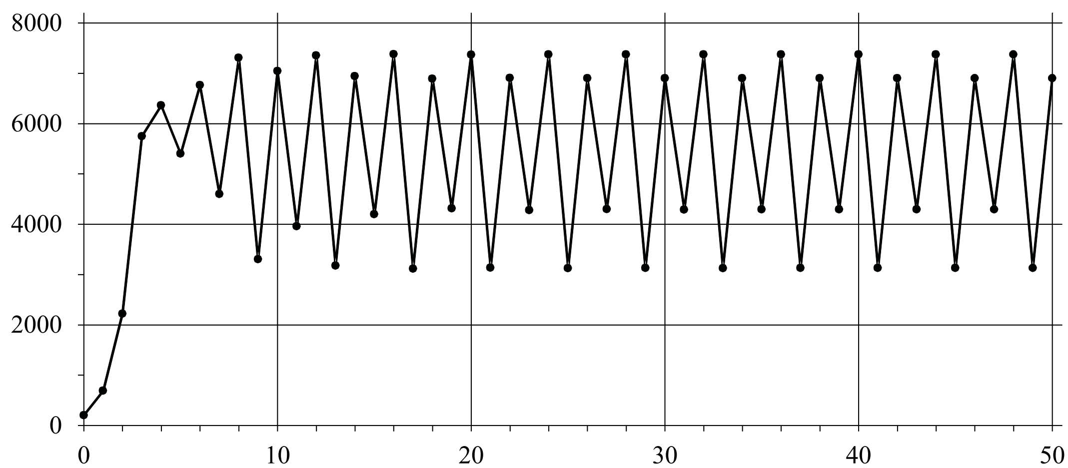

Figure 4: Graph for Example 3.

This version of the model is even more complicated. After an initial rapid growth, the population enters into a cyclic repetition of four distinct values.

Example 4. For our final example we take $r(1000) = 3.4$, while again leaving the equilibrium population size as 6000. This time the difference equation turns out to be

$$p_{n+1} = (3.88 - .00048p_n)p_n,$$

and we obtain the graph shown in Figure 5. Observe that there is no longer a recognizable pattern in the graph. And in fact, this model can be shown to be chaotic.[2]

As these examples show, the discrete logistic growth model exhibits a range of behaviors, depending on the values of the parameters. While it produces a very attractive image of constrained growth in some cases, it can also predict patterns of population growth that are highly unexpected. In a precalculus course emphasizing mathematical modeling, how should we interpret such results?

[2]The results of this model are extremely sensitive to roundoff errors, another characteristic of chaos. A spreadsheet was used to produce Figure 5, computing p_{n+1} as $k(L - p_n)p_n$ with $k = .00048$ and $L = 3.88/k$. Using instead $p_{n+1} = (kL - kp_n)p_n$ produced nearly identical results up to about $n = 65$, after which the results were completely different.

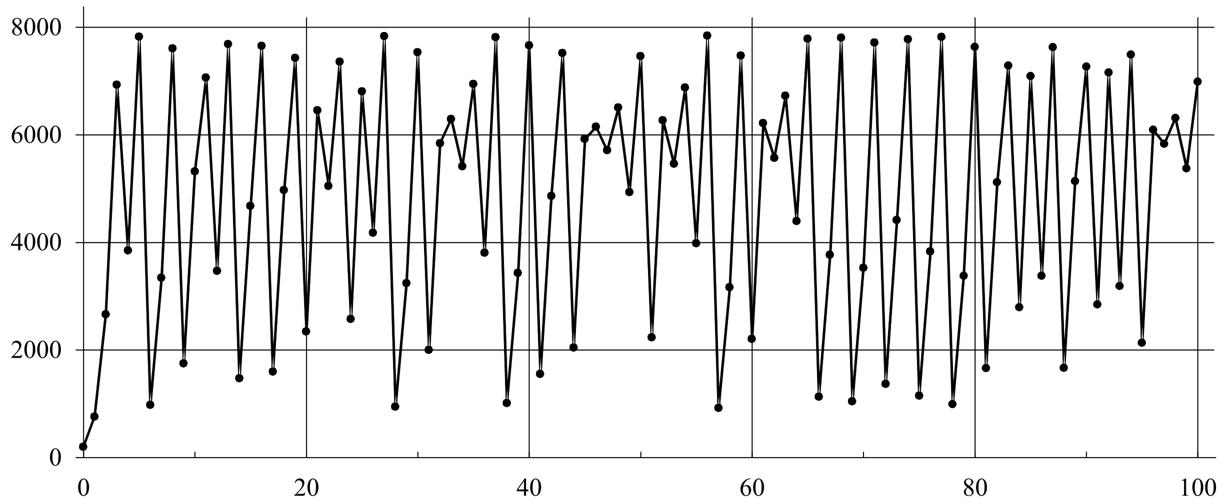

Figure 5: Graph for Example 4.

Logistic Growth as an Example of Chaotic Dynamics. I am not sure who first observed the complicated dynamics that can arise in the discrete logistic model. But Robert May certainly contributed to popularizing the situation with his landmark 1976 paper in *Nature* [16]. As explained in [8], this was during the period when the study of chaos and dynamical systems was emerging as a major new focus of mathematics and mathematical modeling. The May article provides a detailed account of the dynamics of the logistic difference equation, formulated as

$$X_{t+1} = aX_t(1 - X_t),$$

and referred to as equation (3). The article also clearly states May's objectives, including what he described as an *"evangelical plea"* for exposing students to these ideas. At the end of the article, May lamented the way instruction in mathematics and physics was dominated by the study of fundamentally linear problems, cautioning that the

> *... mathematical intuition so developed ill equips the student to confront the bizarre behaviour exhibited by the simplest of discrete nonlinear systems, such as equation (3). Yet such nonlinear systems are surely the rule, not the exception, outside the physical sciences.*
>
> *I would therefore urge that people be introduced to, say, equation (3) early in their mathematical education. ... Such study would greatly enrich the student's intuition about nonlinear systems.*

Given May's goal of illuminating complicated dynamics that can occur in nonlinear systems, his choice of the discrete logistic difference equation is not surprising. On the other hand, if the goal is to develop a realistic model of population growth, one might reasonably ask whether such behavior is inevitable. In other words, is it a feature of the way populations actually grow, or an artifact of the model? As we now demonstrate, the discrete logistic model includes a highly questionable assumption. Moreover, when we adopt a more plausible alternative assumption, the complicated dynamics disappear.

Revised Discrete Logistic Growth

The basic premise of the logistic equation is that the relative growth rate r of a population should decrease as the population size p increases. That is perfectly reasonable. However, we have advanced no justification for modeling the dependence of r on p as a linear function. Indeed, although a linear model is certainly one of the simplest, there is good reason to question it in this context: it predicts that once a population grows above a critical threshhold, the population size will become *negative*.

As a specific instance, consider the difference equation for Example 1:

$$p_{n+1} = .0001(15000 - p_n)p_n.$$

If somehow the population could reach a value of $p_n = 16000$ for some n, the model predicts the next term to be $p_{n+1} = -1600$. For a population model, this is clearly nonsensical.

For this example it can be shown that p_n can never reach 16000 from an initial population p_0 in the interval $(0, 15000)$. But in some other examples, negative populations can arise. Even if the model constrains the population to avoid negative values, assuming a linear equation for $r(p)$ can still be questioned. Is the constant slope of the linear model reasonable? Might it contribute to the counter-intuitive dynamics observable in the discrete logistic model?

Addressing these issues visually, let us contrast the linear model for r with an alternative, as shown in Figure 6. On the left r is a linear function of p, decreasing with a constant slope before crossing the p axis and assuming negative values. On the right, the graph of r flattens out as it approaches the p axis, thus avoiding negative r values. If we define r in that way, what sort of population model would ensue?

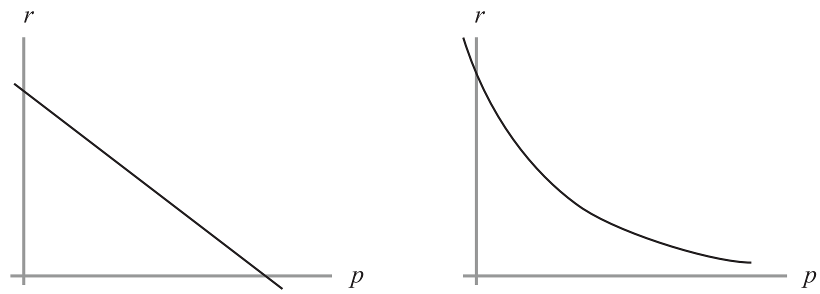

Figure 6: Two approaches to modeling r as a function of p.

Again opting for the greatest algebraic simplicity, we consider an equation of the form

$$r(p) = \frac{1}{mp + b}, \tag{5}$$

with constant m and b. That is, instead of assuming the $r(p)$ is a linear function, we assume it is the reciprocal of a linear function. To put it another way, in recursively generating the terms of our sequence, instead of multiplying each p_n by a linear factor, we will instead *divide* by one.

Taking r as in (5), we obtain the difference equation

$$p_{n+1} = \frac{p_n}{mp_n + b}. \tag{6}$$

We will call this the *revised logistic growth difference equation*; it defines what we will call *revised discrete logistic growth.*

We assume m and b are positive, so that $r(p)$ will be defined and positive for all $p \geq 0$. Notice that the r intercept, $1/b$, is an upper bound for r valid for all $p \geq 0$. If $1/b \leq 1$, the population model will be a strictly decreasing sequence. So, if we wish the population size to experience initial growth before eventually leveling off, we should require $b < 1$. Note also that $r = 1$ when $p = \frac{1-b}{m}$. This corresponds to an equilibrium population size, and will be positive when $b < 1$.

Assuming that r depends on p according to (5), we now revisit the earlier examples, carrying out essentially the same analysis as before.

Revised Example 1. We begin with the same defining assumptions that we used the first time, namely, that $r(1000) = 1.4$ and $r(5000) = 1$. However, now we must find an equation of the form of (5). The nature of this equation suggests that we first find a linear equation for $1/r$. That is, we find $f(p) = 1/r(p) = mp + b$ subject to $f(1000) = 1/1.4$ and $f(5000) = 1$. The result is

$$f(p) = \frac{1}{14000}\, p + \frac{9}{14}.$$

This leads to $r(p) = \frac{1}{f(p)} = \frac{14000}{p+9000}$, and hence our population model has difference equation

$$p_{n+1} = \frac{14000 p_n}{p_n + 9000}.$$

This is equivalent in form to (6), and hence the sequence it generates is an instance of revised discrete logistic growth.

As in the discussion of Example 1, we take an initial value $p_0 = 200$ and apply the difference equation to calculate quite a few terms p_n. The graph of the results is almost identical to what we found in Example 1. This is shown in Figure 7.

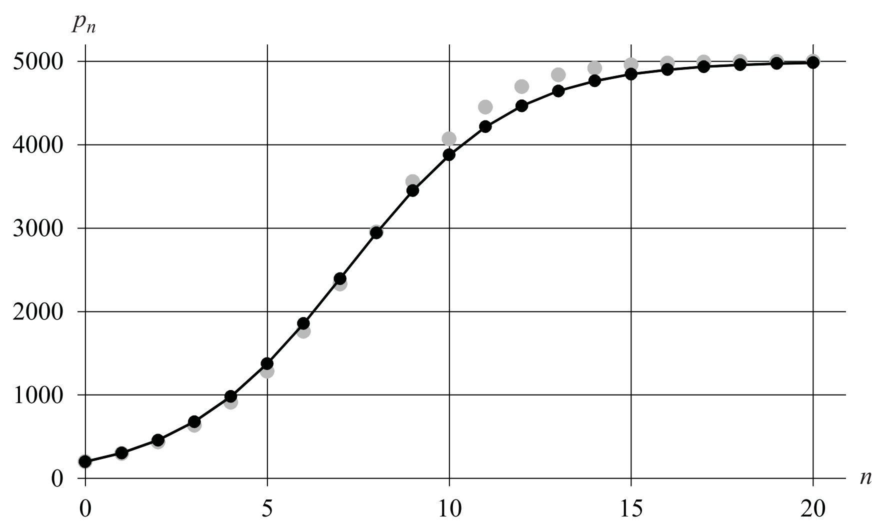

Figure 7: Graph for Revised Example 1. The black dots make up the graph of the revised model, connected with straight lines for visual emphasis. The graph for the original Example 1 is shown with gray dots.

Revised Examples 2 - 4. In a similar way, we can revise examples two through four. In each case, we retain the specified points (p, r) originally used to define r as a linear function of p, but now we use them to define r as a function of p according to (5). The coefficients m and b for each example, rounded to six significant figures, are shown in the table below.

Example	m	b
2	.000125926	.244444
3	.000135484	.187097
4	.000141176	.152941

For each example, we now produce a sequence using the difference equation $p_{n+1} = p_n/(mp_n + b)$. The results are shown grapically in Figure 8.

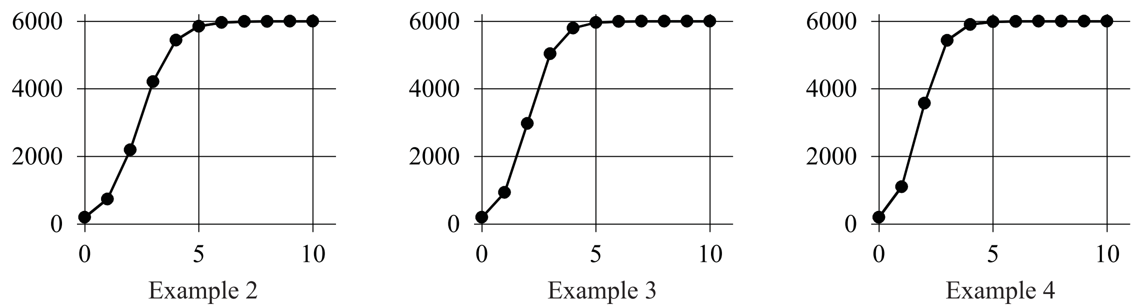

Figure 8: Graphs for Revised Examples 2 - 4.

The graphs for these three revised models are very similar, differing mainly in how rapidly they approach the equilibrium value of 6000. As these examples suggest, and as we shall see next, the complicated dynamics observed in the discrete logistic model do not arise in the revised discrete logistic model.

Solving the revised Difference Equation

In light of the complicated dynamics we saw in Examples 2 - 4, it is not surprising that the discrete logistic growth difference equation has no general closed form solution. The revised equation appears to be algebraically more complicated, at least at first inspection. Seemingly, we should not expect there to be a general solution for the revised equation either. But this intuition is incorrect.

From the dynamicist's perspective, both difference equations are instances of function iteration. A solution is generated by starting with an initial value, p_0, and then repeatedly applying a function f. The terms of the sequence are $p_0, f(p_0), f(f(p_0))$, and so on. The general term is $p_n = f^{(n)}(p_0)$, where $f^{(n)}$ denotes the nth iterate of f. For the discrete logistic growth difference equation (1), $f(p) = k(L - p)p$ is a quadratic polynomial, whereas for the revised equation $f(p) = p/(mp+b)$ it is a fractional linear function. And as the dynamicists know, iterating a quadratic leads to much greater complexity than iterating a fractional linear function. Indeed, it is easy to see that the nth iterate of a quadratic will be a polynomial of degree $2n$, while the nth iterate of a fractional linear function will be another fractional linear function.

Even without the benefit of a dynamicist's perspective, however, the revised equation is solvable by elementary methods. Perhaps the simplest solution follows from a change of variables. We are again motivated to consider reciprocals, this time by inverting both sides of

$$p_{n+1} = \frac{p_n}{mp_n + b}.$$

That produces

$$\frac{1}{p_{n+1}} = \frac{mp_n + b}{p_n} = m + b \cdot \frac{1}{p_n} \ .$$

Thus, if we define $q_n = 1/p_n$, we obtain the linear difference equation

$$q_{n+1} = bq_n + m,$$

with b and m constants. The solution,

$$q_n = \left(q_0 - \frac{m}{1-b}\right) b^n + \frac{m}{1-b}$$

is readily discovered inductively, and leads to an explicit equation for p_n. This method is one instance of a general procedure for solving Ricatti difference equations [17, §6.3].

For an audience of students at the level of college algebra, this approach has a couple of drawbacks. First, it depends on a clever initial step that may be difficult to motivate. On the other hand, the way we inverted r values to find coefficients m and b may at least provide a foundation for a similar approach here.

Second, solving the transformed difference equation is somewhat involved, unless solutions to such equations have already been covered. Accordingly, using the transformation can lead to a nontrivial problem-within-a-problem digression, making the entire argument more difficult to understand.

A more straightforward alternative is to apply an inductive approach directly to the original difference equation. With

$$f(p) = \frac{p}{mp + b,}$$

we can find the second iterate

$$f(f(p)) = \frac{\frac{p}{mp+b}}{\frac{mp}{mp+b} + b},$$

and simplify to

$$\frac{p}{mp + b(mp + b)} = \frac{p}{m(1 + b)p + b^2}.$$

Successive iterates can be similarly computed, and before long a pattern emerges, though the algebra is a little tedious.

For many students in the audience I have in mind, it may be that this derivation is too complicated to follow in detail. But it should be feasible to get across the main ideas – that repeated use of the difference equation leads to a pattern, and algebraic simplification ultimately produces a compact equation for p_n as a function of n. For example, the instructor might derive in detail the equation above for the second iterate, and then simply state that similar methods lead to the results

$$\begin{aligned} f(f(f(p))) &= \frac{p}{m(1 + b + b^2)p + b^3} \\ f(f(f(f(p)))) &= \frac{p}{m(1 + b + b^2 + b^3)p + b^4} \end{aligned} ,$$

and so on. Then students should be able to see what the pattern is, and understand how it can lead to a solution p_n, particularly if they have studied geometric sums.

Happily, for readers of this article, there is a slick alternate solution using matrix algebra. This is obviously not accessible for most precalculus students, but with my strong affinity for matrix methods, I cannot resist sharing the details here.

To begin, let us apply f to a fraction:

$$f\left(\frac{u}{v}\right) = \frac{\frac{u}{v}}{\frac{mu}{v} + b} = \frac{u}{mu + bv}.$$

Thus we see that f imposes the transformation

$$\frac{u}{v} \to \frac{u}{mu + bv}.$$

Notice how closely that resembles a vector operation

$$\begin{bmatrix} u \\ v \end{bmatrix} \to \begin{bmatrix} u \\ mu + bv \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ m & b \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

Applying this repeatedly, if we define

$$\begin{bmatrix} u_n \\ v_n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ m & b \end{bmatrix}^n \begin{bmatrix} u \\ v \end{bmatrix},$$

then

$$f^{(n)}\left(\frac{u}{v}\right) = \frac{u_n}{v_n}.$$

Thus, we can analyze the iterates of f using the powers of the matrix

$$M = \begin{bmatrix} 1 & 0 \\ m & b \end{bmatrix}.$$

Looking at the first few powers of M we find

$$\begin{aligned} M^2 &= \begin{bmatrix} 1 & 0 \\ m & b \end{bmatrix} \begin{bmatrix} 1 & 0 \\ m & b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ m(1+b) & b^2 \end{bmatrix} \\ M^3 &= \begin{bmatrix} 1 & 0 \\ m & b \end{bmatrix} \begin{bmatrix} 1 & 0 \\ m(1+b) & b^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ m(1+b+b^2) & b^3 \end{bmatrix} \\ M^4 &= \begin{bmatrix} 1 & 0 \\ m & b \end{bmatrix} \begin{bmatrix} 1 & 0 \\ m(1+b+b^2) & b^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ m(1+b+b^2+b^3) & b^4 \end{bmatrix}. \end{aligned}$$

As these examples suggest, and as induction easily confirms,

$$M^n = \begin{bmatrix} 1 & 0 \\ m\frac{1-b^n}{1-b} & b^n \end{bmatrix}.$$

Now we can compute $f^{(n)}(p_0)$. First, represent p_0 as a fraction $p_0/1$. Then, with

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} p_0 \\ 1 \end{bmatrix}$$

and

$$\begin{bmatrix} u_n \\ v_n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ m & b \end{bmatrix}^n \begin{bmatrix} p_0 \\ 1 \end{bmatrix},$$

we know $f^{(n)}(p_0) = u_n/v_n$. We proceed to compute

$$\begin{bmatrix} u_n \\ v_n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ m\frac{1-b^n}{1-b} & b^n \end{bmatrix} \begin{bmatrix} p_0 \\ 1 \end{bmatrix} = \begin{bmatrix} p_0 \\ mp_0\frac{1-b^n}{1-b} + b^n \end{bmatrix}.$$

This shows that

$$p_n = f^{(n)}(p_0) = \frac{p_0}{mp_0\frac{1-b^n}{1-b} + b^n}.$$

Further algebraic manipulation leads to

$$p_n = \frac{\frac{1-b}{m}}{1 + \left(\frac{1-b}{mp_0} - 1\right) b^n}. \tag{7}$$

Admittedly, this looks a bit complicated, but observe that it is in the form of the logistic function (2). It becomes simpler if we introduce the parameter $A = (1-b)/m$. Then we have

$$p_n = \frac{A}{1 + \left(\frac{A}{p_0} - 1\right) b^n}.$$

Here, we recognize that A is the limiting value as $n \to \infty$. This value can also be derived as the equilibrium population, where $r(p) = 1$. The remaining logistic function parameter, $B = A/p_0 - 1$, also has a natural interpretation. Write it as $B = (A - p_0)/p_0$, and consider A as the sustainable population size. Then the numerator tells us the initial capacity for population growth, and B expresses this as a percentage. Thus, if $B = .50$, the initial population can increase by 50%; if $B = 4$, the initial population can increase by 400%, that is, to $5p_0$.

The solution (7) shows that $p_n = P(n)$ where $P(t)$ is the continuous logistic growth function (2), with $A = (1-b)/m$, and $B = A/p_0 - 1$. Thus, if we graph the points (n, p_n), they fall on a continuous logistic growth curve. This shows, in a sense, that the revised equation is the *right* discrete analog of the logistic growth differential equation.

In fact, this points to an alternative derivation of the revised difference equation. Define a sequence p_n by sampling a continuous logistic growth function $P(t)$ at regularly spaced values of t, and determine how each term depends on its predecessor. Historically, this is probably how (6) was first derived, accounting for fact that itis sometimes called the *Verhulst difference equation*. More will be said on the history later.

Returning to our analysis of the difference equation (6), the solution (7) leads to an understanding of the behavior of the sequence. If $p_0 < A$, so that the initial term is less than the equilibrium value, then $B > 0$ and p_n increases monotonically toward the equilibrium. Otherwise, with p_0 above the equilibrium value, $B < 0$ and p_n decreases monotonically toward the equilibrium. These results demonstrate that complicated dynamics can never arise in a model governed by the revised logistic difference equation: the graph of the sequence must lie on a continuous logistic growth curve. Thus our goal in modifying (1) has been achieved.

Interestingly, it turns out that in some cases a model based on (1) may be more faithful to actual populations than the revised model. As one example, Utida [20] studied insect populations where each generation matures, reproduces, and dies in a common 25 day period. He found that the oscillatory behavior characteristic of the discrete logistic model is observable in nature. Thus, for this population, the revised difference equation is disqualified by the very aspect that we set out to obtain, its avoidance of complicated dynamics. And this leads naturally to the discussion of another important aspect of model development: validating the predictions of a model by observing an actual system. The existence of work like Utida's further enriches the topic of discrete logistic growth as an illustration of mathematical modeling.

Additional Remarks

There remain several aspects of this topic that are worthy of consideration, though we can only mention them briefly here.

Möbius Transformations. First, the appearance of the matrix M above is associated with the more general concept of a Möbius transformation. This is any function

$$f(t) = \frac{At + B}{Ct + D}$$

defined either for a real or complex variable t. Corresponding to f we have the matrix

$$M_f = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

If f and g are both Möbius transformations, then so is their composition $f \circ g$ and $M_{f \circ g} = M_f M_g$. In particular $M_{f^{(n)}} = (M_f)^n$, as we found in the special case we considered earlier.

The subject of Möbius transformations is broad and deep, connecting in important ways with complex analysis, geometry, dynamical systems, and other topics.

Matrix Powers. For our specific application, the simple form of the matrix M allowed us to discover an equation for M^n. This problem can also be approached by diagonalizing the matrix M. Because our M is triangular, we can read off the eigenvalues 1 and b, and diagonalization proceeds quite easily. Something similar happens with the difference equation $p_{n+1} = f(p_n)$ with $f(p) = bp + m$. That is also a Möbius transformation, this time with matrix

$$M = \begin{bmatrix} b & m \\ 0 & 1 \end{bmatrix},$$

which is again triangular. Using this fact, it is a pleasant exercise to derive the solution to the general first order linear difference equation $p_{n+1} = bp_n + m$. Comparing the two derivations shows that dynamically, there is essentially no difference between the revised difference equation and the apparently simpler $p_{n+1} = bp_n + m$.

Harvesting and Cobweb Diagrams. One variant on the logistic growth model involves inclusion of a term representing harvesting. For example, if $p_{n+1} = f(p_n)$ models a population in a state of nature, then $p_{n+1} = f(p_n) - h$ can be used to model the effect of harvesting or consuming h members of the population per iterative cycle. This is discussed in conjunction with (1) in [12], among other sources. Applying similar methods to the revised model leads to similar, but somewhat simpler, results.

These models are effectively studied using cobweb diagrams, a nice way to visualize function iteration. The cobweb diagram for the sequence $x_n = f^{(n)}(x_0)$ is a rectangular path defined by the points $(x_0, x_1), (x_1, x_1), (x_1, x_2), (x_2, x_2), \cdots$. These alternate between points on the graph G of f and points on the line L for which $y = x$. The diagram is generated by alternately moving vertically from L to G and horizontally from G to L. An example for a quadratic function f is shown in Figure 9. Cobweb diagrams can be used to show why chaos arises in the discrete logistic model, but not in the revised discrete logistic model. They also offer a simple way to analyze the effect of harvesting in both the original and revised discrete logistic models.

A little history. Among teachers and curriculum developers for precalculus mathematics, *discrete logistic growth* and *logistic difference equation* are invariably associated with models defined by

$$p_{n+1} = k(L - p_n)p_n$$

or the equivalent. That is certainly the supposition in [4, 10, 14, 18]. However, that is not true in the more specialized literature on mathematical models in population biology,

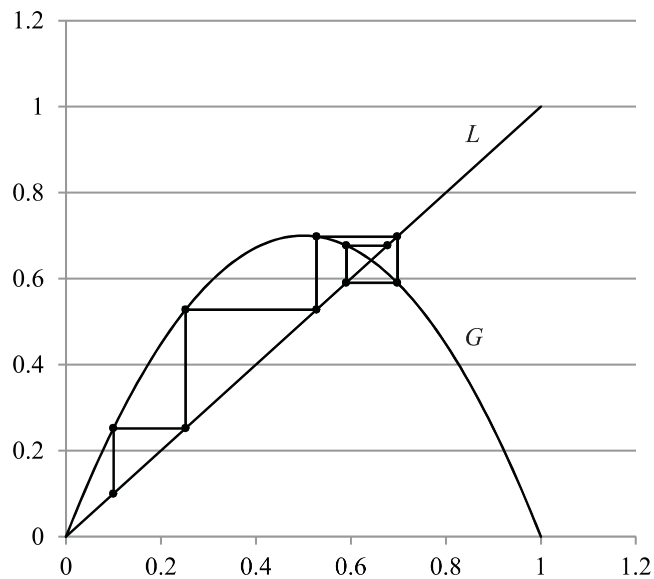

Figure 9: Cobweb diagram for an iteration sequence of a quadratic function f. The initial point is $(x_0, x_0) = (.1, .1)$. Succeeding points are $(x_0, f(x_0)) = (x_0, x_1)$, (x_1, x_1), $(x_1, f(x_1)) = (x_1, x_2)$, (x_2, x_2), etc.

epidemiology, and related fields. There, logistic growth can refer to a variety of different models.

Verhulst is credited with formulating the differential equation

$$p' = k(L - p)p$$

in 1838, and coining the name *logistic* (see [21, 22]). Later investigators proposed variations on Verhulst's equation, sometimes continuing to refer to these as logistic models.

In this same literature, difference equations also appear. In particular,

$$p_{n+1} = \frac{p_n}{mp_n + b}, \tag{8}$$

which I have called the revised logistic difference equation, is well known. It is also well known that the solutions of (8) lie on solution curves to Verhulst's differential equation, thus revealing a perfect analogy between the differential and difference equations.

In the literature, (8) has frequently been referred to as the Verhulst difference equation (see for example [2, 3, 6, 11]), sometimes with an accompanying citation of [21] or [22].

However, I can find no indication in those works that Verhulst considered difference equations. It may be that the difference equation is considered to be a discrete analog of the Verhulst *differential* equation by virtue of their analogous *solutions*. Be that as it may, the association of Verhulst with (8) is well established. May [15] remarks, citing several sources, that (8) "is sometimes called *the* logistic difference equation" (his emphasis). So to these authors at least, discrete logistic growth is associated not with my first difference equation, but with the revised equation.

Equivalent versions of (8) are also referred to as the Beverton-Holt difference equation or the Pielou difference equation. But again I find no indication that these authors were interested in developing discrete population models based on a (constant coefficient) difference equation. In Beverton and Holt's [1], for example, (8) arises in connection with piecewise smooth population models, where each smooth arc is a solution to Verhulst's differential equation whose parameters vary from one arc to another. In particular, it is never used for more than one iteration with a given set of parameters. This combination of continuous and discrete effects is referred to in [2] as a *metered model*.

Beverton and Holt use their version of (8) to determine initial conditions for one arc from the initial conditions of the preceding arc. Thus, they derive the difference equation by relating $f(t+1)$ to $f(t)$ for a logistic function f. In this context, it is evident that the solution to the difference equation will lie on a logistic curve, and this appears to account for the fact that the difference equation and Verhulst's differential equation are known to have analogous solutions. In the papers cited by May, as mentioned above, the anology is generally established by the Beverton/Holt approach.

Based on my admittedly limited review, discrete models defined by (8) are too simple to be of much direct interest in the primary literature. On the other hand, I am not the first to compare such models with those based on (1). May [15] describes my (1) as "probably the simplest nonlinear difference equation one could write down," and notes several authors who have discussed it as the analog of the logistic differential equation. But he also observes that this difference equation suffers from the "unbiological feature that the population can become negative." He mentions several other possible difference equations that avoid this limitation, among them (8). However, May's object is the opposite of mine. Where I wish to develop a model that avoids the complicated dynamics of (1), he proceeds to analyze these complicated dynamics in detail.

I am also not the first to derive the solution to (8) directly, as opposed to sampling points from a logistic curve. Indeed, there are significant similarities between the analysis in this article and the treatments in [4, 11]. If I have contributed anything new, it is the idea of starting with discrete logistic growth, and by reconsidering the assumptions of the model, "discovering" continuous logistic functions. This makes sense pedagogically for a precalculus audience not familiar with logistic functions.

Conclusion

I like difference equations because they can make aspects of modeling accessible to students who have not studied calculus. In particular, I hope students will get an idea of how models can be formulated, investigated and refined. Toward this end the progression of topics discussed in this paper has much to offer. There is an understandable and plausible succession of models, from unbounded geometric growth, to the discrete logistic equation, to the revised equation. We can explore the role of parameters in difference equations, investigating the behavior of solutions numerically, graphically, and analytically. And at the end, we have a dramatic opportunity to stress the importance of validation through direct observation of the system being modeled. Appropriating May's injunction, I would therefore urge that students be introduced to developments of this sort as part of their general educations. Such study would greatly enrich the student's understanding of how mathematics is applied through the creation and refinement of mathematical models.

Acknowledgments: I thank Robert Devaney for pointing out the connection between Möbius transformations and the revised logistic difference equation. I also thank Fred Brauer for generously helping me learn a little about population models and their history.

References

[1] R. J. H. Beverton, Sidney J. Holt, *On the Dynamics of Exploited Fish Populations.* H. M. Stationery Office, London, 1957.

[2] Fred Brauer, Carlos Castillo-Chavez, *Mathematical Models in Population Biology and Epidemiology.* Springer, New York, 2012.

[3] Leonid Berezansky, Elena Braverman, On impulsive Beverton-Holt difference equations and their applications *Journal of Difference Equations and Applications* **10** no. 9, 10 (2004) 851-868, http://dx.doi.org/10.1080/10236190410001726421.

[4] James L. Cornette and Ralph A. Ackerman, *Calculus for the Life Sciences: A Modeling Approach* (e-book), Mathematical Association of America, Wachington, DC, 2015.

[5] Bruce Crauder, Benny Evans, Alan Noell, *Functions and Change: A Modeling Approach to College Algebra,* 5th edition. Cengage Learning, Belmont, CA, 2013.

[6] M. De la Sen, The generalized Beverton-Holt equation and the control of populations, *Applied Mathematical Modelling* **32** (2008) 2312–2328, http://dx.doi.org/10.1016/j.apm.2007.09.007.

[7] Franklin Demana, Bert K. Waits, Gregory D. Foley, Daniel Kennedy, *Precalculus: Functions and Graphs,* 4th edition. Addison Wesley, Reading, MA, 2001.

[8] James Gleick, *Chaos: Making a New Science.* Viking, New York, 1987.

[9] Sheldon P. Gordon, Florence S. Gordon, Alan C. Tucker, Martha J. Siegel, *Functioning in the Real World: A Precalculus Experience,* 2nd edition.Addison Wesley, Boston, 2004.

[10] Sheldon P. Gordon, Comparing the discrete and continuous logistic models, *Primus* **18** no. 5 (2008) 449–455.

[11] Frank C. Hoppensteadt, *Mathematical Methods of Population Biology.* Cambridge University Press, Cambridge, Cambridgeshire, 1982.

[12] Dan Kalman, *Elementary Mathematical Models: Order Aplenty and a Glimpse of Chaos.* Mathematical Association of America, Washington, DC, 1997.

[13] Ron Larson, David C Falvo, *College Algebra,* 9th edition. Cengage, Boston, 2014.

[14] Joseph M. Mahaffy, *Logistic Growth and Nonlinear Dynamical Systems* (webpage), http://www-rohan.sdsu.edu/~jmahaffy/courses/s00/math121/lectures/logistic_growth/logistic.html, accessed 7/6/2016.

[15] Robert M. May, Biological populations obeying difference equations: stable points, stable cycles, and chaos, *J. theor. Biol.* **51** (1975) 511–524, http://www.sciencedirect.com/science/article/pii/0022519375900788.

[16] —, Simple mathematical models with very complicated dynamics, *Nature* **261** (1976) 459–467, http://isites.harvard.edu/fs/docs/icb.topic1313269.files/Lecture22/May_Simple

[17] R. Mickens, *Difference Equations, 2nd Edition*, CRC Press/Chapman and Hall, Boca Raton, FL, 1991.

[18] D. Q. Nykamp, J. L. Cornette, R. A. Ackerman, Environmental carrying capacity, *Math Insight* (webpage), http://mathinsight.org/environmental_carrying_capacity, accessed 7/6/2016.

[19] James Stewart, L. Redlin, Saleem Watson, *Precalculus: Mathematics for Calculus,* 6th edition. Cengage Learning, Belmont, CA, 2012.

[20] Syunro Utida, Damped oscillation of population density at equilibrium, *Researches on Population Ecology* **9** No. 1 (1967) 1–9.

[21] P. F. Verhulst, Notice sur la loi que la population suit dans son accroissement, *Corresp. Math. Phys.* **10** (1838) 113–121, https://books.google.fr/books?id=8GsEAAAAYAAJ&printsec=frontcover&source=gbs_v2_summar

[22] —, Recherches mathématiques sur la loi d'accroissement de la population, *Nouveaux Mémoires de l'Académie Royale des Sciences et Belles-Lettres de Bruxelles* **18** (1845) 1–42, http://gdz.sub.uni-goettingen.de/dms/load/img/?PPN=PPN129323640_0018&DMDID=dmdlog7.

[23] Frank Wilson , Scott L. Adamson , Trey Cox , Alan E. O'Bryan, *College Algebra: A Make it Real Approach.* Cengage Learning, Belmont, CA, 2012.